S.C. Kleene Introduction to Metamathematics
North Holland, 2000 (from 1952 impression)

communication of a statement giving an x such that $A(x)$ (Hilbert-Bernays 1934 p. 32). But "$A(x)$" itself may in turn be an incomplete communication. Accordingly let us say that "$(Ex)A(x)$" is an incomplete communication, which is completed by giving an x such that $A(x)$ together with the further information required to complete the communication "$A(x)$" for that x.

The idea can be extended to the other logical operations. For example, we can regard a generality statement "$(x)A(x)$" intuitionistically as an incomplete communication, which is completed by giving an effective general method for finding, to any x, the information which completes the communication "$A(x)$" for that x.

Similarly, an implication "$A \rightarrow B$" can be regarded as an incomplete communication, which is completed by giving an effective general method for obtaining the information which completes "B", whenever that which completes "A" is given.

Negation can be reduced to implication (cf. Example 3 § 74).

Now effective general methods are recursive ones, when it is a natural number that is being given (§§ 60, 62, 63). Moreover, by the device of Gödel numbering, information can be given by a number.

Combining these ideas, we shall define a property of a number-theoretic formula which will amount to the formula's being true under the interpretation suggested. However, instead of saying 'true', we shall say '(recursively) realizable', to distinguish the property defined below from 'truth' as defined by using direct translations of the formal logical symbols by corresponding informal words (end § 81).

The interpretation of a term $t(x_1, \ldots, x_n)$ containing only $x_1, \ldots, x_n$ free by a primitive recursive function $t(x_1, \ldots, x_n)$, or for $n = 0$ by a number t, and the interpretation of a prime formula $P(x_1, \ldots, x_n)$ by a primitive recursive predicate $P(x_1, \ldots, x_n)$, or for $n = 0$ by a proposition P (end § 81), do not differ intuitionistically from classically. We build upon this in setting up the definition of 'realizability' which interprets the logical operators intuitionistically as applied to number-theoretic formulas.

First we define the circumstances under which a natural number e '(recursively) realizes' (or is a 'realization number' of) a closed number-theoretic formula E, by induction on the number of (occurrences of) logical symbols in E.

(A) 1. *e realizes* a closed prime formula P, if $e = 0$ and P is true (in other words, if $e = 0$ and P).

§82 starts on p. 501 * Kleene gives a computational semantics for HA. "The meaning of the existential statement ... is an incomplete communication ...

For Clauses 2 -- 5, A and B are any closed formulas.

2. *e realizes* A & B, if $e = 2^a \cdot 3^b$ where *a realizes* A and *b realizes* B.

3. *e realizes* A ∨ B, if $e = 2^0 \cdot 3^a$ where *a realizes* A, or $e = 2^1 \cdot 3^b$ where *b realizes* B.

4. *e realizes* A ⊃ B, if e is the Gödel number of a partial recursive function φ of one variable such that, whenever *a realizes* A, then $\varphi(a)$ *realizes* B.

5. *e realizes* ¬A, if *e realizes* A ⊃ 1=0.

For Clauses 6 and 7, x is a variable, and A(x) a formula containing free only x.

6. *e realizes* ∃xA(x), if $e = 2^x \cdot 3^a$ where *a realizes* A(**x**).

7. *e realizes* ∀xA(x), if e is the Gödel number of a general recursive function φ of one variable such that, for every x, $\varphi(x)$ *realizes* A(**x**).

Now we define '(recursive) realizability' for any number-theoretic formula, thus.

(B) A formula A containing no free variables is *realizable*, if there exists a number p which realizes A. A formula $A(y_1, \ldots, y_m)$ containing free only the distinct variables $y_1, \ldots, y_m$ ($m \geq 0$) is *realizable*, if there exists a general recursive function φ of m variables (called a *realization function* for $A(y_1, \ldots, y_m)$) such that, for every $y_1, \ldots, y_m$, $\varphi(y_1, \ldots, y_m)$ realizes $A(\mathbf{y}_1, \ldots, \mathbf{y}_m)$. (Using § 44, if a given formula is realizable for one choice of the $y_1, \ldots, y_m$, it is for every other.)

The handling of the free variables in the present definition of realizability differs from that in Kleene 1945. It simplifies the proof of the first theorem (Theorem 62), after which the equivalence of the two definitions will follow (by Corollary 1).

The above definition of realizability refers only to our notion of number-theoretic formula, i.e. to the formation rules of our formal system.

A modified notion of realizability, referring to the postulate list of the system, and to assumption formulas Γ if desired, is obtained by altering three clauses, as follows. Clause 3: replace "*a realizes* A" by "*a realizes* A and Γ ⊢ A", and "*b realizes* B" by "*b realizes* B and Γ ⊢ B". Clause 4: replace "*a realizes* A" by "*a realizes* A and Γ ⊢ A". Clause 6: replace "*a realizes* A(**x**)" by "*a realizes* A(**x**) and Γ ⊢ A(**x**)". For 'realizes' ['realizable'] in this modified sense we say *realizes*-(Γ ⊢) [*realizable*-(Γ ⊢)].

THEOREM 62^{N}. (a) *If* $\Gamma \vdash$ E *in the intuitionistic number-theoretic formal system, and the formulas* Γ *are realizable, then* E *is realizable.* (David Nelson 1947 Part I.)

(b) *Similarly reading "realizable-*($\Gamma \vdash$)*" in place of "realizable".*

LEMMA 44^{N}. *If* x *is a variable,* A(x) *is a formula without free variables other than* x, *and* t *is a term without variables which hence expresses a number* t, *then* e *realizes* A(t) *if and only if* e *realizes* A($\boldsymbol{t}$).

PROOF OF LEMMA 44. If A(x) is prime, then whether A(t) is true is equivalent to whether A($\boldsymbol{t}$) is true. Hence by Clause 1, the lemma holds for a prime A(x). The lemma for any other A(x) follows from this basis by induction on the number of logical symbols in A(x), with cases corresponding to the other clauses in the definition of 'realizes'.

LEMMA 45^{N}. *If* E *is a closed formula, then* e *realizes* E *if and only if* e *realizes the result of replacing each part of* E *of the form* $\neg$A *where* A *is a formula by* A $\supset$ 1=0.

Lemmas 44 and 45 also hold reading "$\Gamma \vdash$" or "*e realizes-*($\Gamma \vdash$)" in place of "*e realizes*", when $\vdash$ refers to the intuitionistic number-theoretic system, and Γ are any formulas. (For Lemma 44 we then use (A) § 41 with Theorem 24 (b) § 38.)

PROOF OF THEOREM 62. We state the proof for (a), and (optionally) the reader, by taking slight extra care, can verify that the additional conditions are met for (b). The proof is by induction on the length of the given deduction $\Gamma \vdash$ E, with cases corresponding to the postulates of our formal system.

First we consider AXIOMS. If A($y_1, \ldots, y_m$) is an axiom containing as its only free variables $y_1, \ldots, y_m$, then by (B) to establish its realizability we must give a general recursive function $\varphi(y_1, \ldots, y_m)$ such that, for every m-tuple of natural numbers $y_1, \ldots, y_m$, the number $\varphi(y_1, \ldots, y_m)$ realizes A($\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$). However, for each of the axiom schemata of the propositional calculus, we shall be able to find a number which realizes A($\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$) for any axiom A($y_1, \ldots, y_m$) by the schema. It will suffice to give this number (which realizes the closed axioms by the schema), because when free variables $y_1, \ldots, y_m$ are present, we can take as $\varphi(y_1, \ldots, y_m)$ the constant function of m variables with this number as value (§ 44). Similarly for the particular number-theoretic axioms, we shall merely give a number which realizes the result of any substitution of numerals for the free variables of the axiom. Similarly for Axiom Schema 13, we can give a realization number, as a general recursive function of x, which depends only on the numeral $\boldsymbol{x}$ substituted for x; and for each of Axiom Schemata 10 and 11 one can be given, as a general recursive function of $x_1, \ldots, x_n$, which depends only on the t and on the numerals $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ substituted for its variables $x_1, \ldots, x_n$. Then when the $y_1, \ldots, y_m$ include other variables, the $\varphi(y_1, \ldots, y_m)$ can be obtained by expanding that function into a function of the required additional variables by use of identity functions (§ 44).

For each of the axiom schemata and particular axioms (§§ 19, 23), we shall express our realization number or function using the notations of § 65. The proof that it is a realization number or function, and the necessary verifications of recursiveness, are left to the reader in cases not discussed in detail.

1a. In accordance with the preliminary remarks, consider an axiom A $\supset$ (B $\supset$ A) by this schema containing no free variables. We show that $\Lambda a \Lambda b\, a$, i.e. $\Lambda a \Lambda b\, U_1^2(a, b)$ (§ 44), realizes A $\supset$ (B $\supset$ A). For let a realize A; by Clause 4, we must show that $\{\Lambda a \Lambda b\, a\}(a)$, i.e. $\Lambda b\, a$ (by (71) § 65), realizes B $\supset$ A. To show this, let b realize B; we must show that $\{\Lambda b\, a\}(b)$, i.e. a, realizes A. But a does realize A, by hypothesis.

1b. (A $\supset$ B) $\supset$ ((A $\supset$ (B $\supset$ C)) $\supset$ (A $\supset$ C)) is realized by $\Lambda p \Lambda q \Lambda a\, \{q(a)\}(p(a))$. For let p realize A $\supset$ B; we must show that $\{\Lambda p \Lambda q \Lambda a\, \{q(a)\}(p(a))\}(p)$, i.e. $\Lambda q \Lambda a\, \{q(a)\}(p(a))$, realizes (A $\supset$ (B $\supset$ C)) $\supset$ (A $\supset$ C). To show this, let q realize A $\supset$ (B $\supset$ C); we must show that $\Lambda a\, \{q(a)\}(p(a))$ realizes A $\supset$ C. To show this, let a realize A; we must show that $\{q(a)\}(p(a))$ realizes C. Now by hypothesis, p realizes A $\supset$ B and a realizes A; hence $p(a)$ realizes B. Moreover q realizes A $\supset$ (B $\supset$ C), and a realizes A; so $q(a)$ realizes B $\supset$ C. But now $q(a)$ realizes B $\supset$ C, and $p(a)$ realizes B; hence $\{q(a)\}(p(a))$ realizes C, as was to be shown.

3. A $\supset$ (B $\supset$ A & B). $\Lambda a \Lambda b\, 2^a \cdot 3^b$.

4a. A & B $\supset$ A. $\Lambda c\, (c)_0$ (cf. #19 § 45). 4b. A & B $\supset$ B. $\Lambda c\, (c)_1$.

5a. A $\supset$ A $\vee$ B. $\Lambda a\, 2^0 \cdot 3^a$. 5b. B $\supset$ A $\vee$ B. $\Lambda b\, 2^1 \cdot 3^b$.

6. (A $\supset$ C) $\supset$ ((B $\supset$ C) $\supset$ (A $\vee$ B $\supset$ C)).

$\Lambda p \Lambda q \Lambda r\ \chi(p, q, r)$ where

$\chi(p, q, r) \simeq [p((r)_1)$ if $(r)_0 = 0$, $q((r)_1)$ if $(r)_0 = 1]$, using Theorem XX (c). Suppose p realizes A $\supset$ C, q realizes B $\supset$ C, and r realizes A $\vee$ B; we must show that $\chi(p, q, r)$ realizes C. CASE 1: $r = 2^0 \cdot 3^a$ where a realizes A. Then $(r)_0 = 0$ and $(r)_1 = a$. Since p realizes A $\supset$ C and $(r)_1$ realizes A, $p((r)_1)$ realizes C. But $(r)_0 = 0$; so (and because $p((r)_1)$ is defined) $\chi(p, q, r) = p((r)_1)$, and so it realizes C, as was to be shown. CASE 2: $r = 2^1 \cdot 3^b$ where b realizes B. Similarly.

7. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$. Using Lemma 45, the number which realizes the closed axioms by Axiom Schema 1b (in particular those with $1=0$ as the C) realizes those by this schema.

8^{I}. $\neg A \supset (A \supset B)$. 0. For if p realizes $\neg A$, then by Clause 5, p realizes $A \supset 1=0$. But then no number a can realize A, since $p(a)$ would realize the false closed prime formula $1=0$, contradicting Clause 1. Thus vacuously, if p realizes $\neg A$ and a realizes A, then $\{0(p)\}(a)$ realizes B.

(The reader may find it instructive to verify that there is no apparent way to treat the classical Axiom Schema 8.)

10. Let the t for the axiom contain exactly the distinct variables $x_1, \ldots, x_n$ $(n \geq 0)$; denote it as "$t(x_1, \ldots, x_n)$", and let $t(x_1, \ldots, x_n)$ be the primitive recursive function (or for $n = 0$, the number) which it expresses. By the preliminary remarks, we suppose the axiom contains free only $x_1, \ldots, x_n$; if none of $x_1, \ldots, x_n$ is x, let it be $\forall x A(x, x_1, \ldots, x_n) \supset A(t(x_1, \ldots, x_n), x_1, \ldots, x_n)$. Since $t(x_1, \ldots, x_n)$ is free for x in $A(x, x_1, \ldots, x_n)$, the result of substituting numerals $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ for (the free occurrences of) $x_1, \ldots, x_n$ in the axiom is $\forall x A(x, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \supset A(t(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. We shall show that the number $\Lambda p\, p(t(x_1, \ldots, x_n))$, which as $x_1, \ldots, x_n$ vary is a general (in fact, primitive) recursive function of $x_1, \ldots, x_n$, realizes this formula. By Clause 4, for this purpose we must show that, if p realizes $\forall x A(x, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, then $p(t(x_1, \ldots, x_n))$ realizes $A(t(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. But, if p realizes $\forall x A(x, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, then by Clause 7, $p(t(x_1, \ldots, x_n))$ realizes $A(\boldsymbol{t}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ where $t = t(x_1, \ldots, x_n)$; and hence by Lemma 44, $p(t(x_1, \ldots, x_n))$ also realizes $A(t(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. — If say x_1 is x, the axiom is $\forall x_1 A(x_1, \ldots, x_n) \supset A(t(x_1, \ldots, x_n), x_2, \ldots, x_n)$, etc.

11. $A(t(x_1, \ldots, x_n), x_1, \ldots, x_n) \supset \exists x A(x, x_1, \ldots, x_n)$. $\Lambda a\, 2^{t(x_1, \ldots, x_n)} \cdot 3^{a}$.

13. $A(0) \,\&\, \forall x(A(x) \supset A(x')) \supset A(x)$. We treat the case that the A(x) contains free only x, as the preliminary remarks will then take care of the general case. Let a partial recursive function $\rho(x, a)$ be defined by a primitive recursion thus,

$$\begin{cases} \rho(0, a) = (a)_0, \\ \rho(x', a) \simeq \{\{(a)_1\}(x)\}(\rho(x, a)). \end{cases}$$

Now we show that for every x the number $\Lambda a\, \rho(x, a)$, which is a primitive recursive function of x, realizes $A(0) \,\&\, \forall x(A(x) \supset A(x')) \supset A(\boldsymbol{x})$. To do so (Clause 4), we prove by induction on x that, if a realizes $A(0) \,\&\, \forall x(A(x) \supset A(x'))$, then $\rho(x, a)$ realizes $A(\boldsymbol{x})$. BASIS. If a realizes $A(0) \,\&\, \forall x(A(x) \supset A(x'))$, then by Clause 2, $\rho(0, a)$ $[= (a)_0]$ realizes A(0). IND. STEP. Similarly $(a)_1$ realizes $\forall x(A(x) \supset A(x'))$, and hence (Clause 7) $\{(a)_1\}(x)$ realizes $A(\boldsymbol{x}) \supset A(\boldsymbol{x}')$. But by hyp. ind., $\rho(x, a)$ realizes $A(\boldsymbol{x})$. Hence (Clause 4), $\rho(x', a)$ $[= \{\{(a)_1\}(x)\}(\rho(x, a))]$ realizes $A(\boldsymbol{x}')$.

14. After substitution of numerals, we have from this axiom $\boldsymbol{a}'=\boldsymbol{b}' \supset \boldsymbol{a}=\boldsymbol{b}$. This formula is realized by Λp 0. For suppose p realizes $\boldsymbol{a}'=\boldsymbol{b}'$. We must show that then 0 realizes $\boldsymbol{a}=\boldsymbol{b}$. Since $\boldsymbol{a}'=\boldsymbol{b}'$ is prime, it is only realizable if it is true, i.e. if $a' = b'$. Then $a = b$, so $\boldsymbol{a}=\boldsymbol{b}$ is also true, and 0 realizes it.

Similarly, for the other particular axioms, after substituting numerals, we have realization numbers as follows.

15, 18 — 21: 0. 16: $\Lambda p \Lambda q$ 0. 17: Λp 0.

RULES OF INFERENCE. 2. We take advantage of the remark accompanying the definition of realizability to regard the formulas as each dependent on all of the variables occurring free in any of them. Thus we write the rule

$$\frac{A(y_1, \ldots, y_m) \qquad A(y_1, \ldots, y_m) \supset B(y_1, \ldots, y_m)}{B(y_1, \ldots, y_m).}$$

By hypothesis of the induction, the premises $A(y_1, \ldots, y_m)$ and $A(y_1, \ldots, y_m) \supset B(y_1, \ldots, y_m)$ are realizable, i.e. there are general recursive functions α and ψ, such that, for every m-tuple of natural numbers $y_1, \ldots, y_m$, $A(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$ is realized by the number $\alpha(y_1, \ldots, y_m)$ and $A(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) \supset B(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$ by the number $\psi(y_1, \ldots, y_m)$. Then the number $\{\psi(y_1, \ldots, y_m)\}(\alpha(y_1, \ldots, y_m))$ realizes $B(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$. Moreover, $\{\psi(y_1, \ldots, y_m)\}(\alpha(y_1, \ldots, y_m))$ is obviously a partial recursive function of $y_1, \ldots, y_m$. But its value is a realization number for every $y_1, \ldots, y_m$, so it must be defined for every $y_1, \ldots, y_m$; thus it is general recursive. Thus the conclusion $B(y_1, \ldots, y_m)$ is realizable.

9. $$\frac{C(y_1, \ldots, y_m) \supset A(x, y_1, \ldots, y_m)}{C(y_1, \ldots, y_m) \supset \forall x A(x, y_1, \ldots, y_m).}$$

By the hypothesis of the induction and the definition of realizability, there is a general recursive function ψ such that, for every $x, y_1, \ldots, y_m$, $\psi(x, y_1, \ldots, y_m)$ realizes $C(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) \supset A(\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$. We shall prove that, for every $y_1, \ldots, y_m$, $\Lambda c \Lambda x \{\psi(x, y_1, \ldots, y_m)\}(c)$ realizes $C(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) \supset \forall x A(x, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$. This will give the realizability of the conclusion, since $\Lambda c \Lambda x \{\psi(x, y_1, \ldots, y_m)\}(c)$ is a primitive recursive, a fortiori general recursive, function of $y_1, \ldots, y_m$. Accordingly suppose that

c realizes $C(y_1, \ldots, y_m)$; we must show that $\Lambda x \{\psi(x, y_1, \ldots, y_m)\}(c)$ realizes $\forall x A(x, y_1, \ldots, y_m)$. To do this, we must show that, for every x, $\{\psi(x, y_1, \ldots, y_m)\}(c)$ realizes $A(x, y_1, \ldots, y_m)$. But since c realizes $C(y_1, \ldots, y_m)$, and by hyp. ind. $\psi(x, y_1, \ldots, y_m)$ realizes $C(y_1, \ldots, y_m) \supset A(x, y_1, \ldots, y_m)$, $\{\psi(x, y_1, \ldots, y_m)\}(c)$ does realize $A(x, y_1, \ldots, y_m)$. (Note how this treatment would break down, if the C contained x free, call it "$C(x, y_1, \ldots, y_m)$". Then, we would have to assume that c realizes $C(x, y_1, \ldots, y_m)$ for some x, and we could conclude only that $\{\psi(x, y_1, \ldots, y_m)\}(c)$ realizes $A(x, y_1, \ldots, y_m)$ for that x, whereas we would need to conclude it for every x.)

12.
$$\frac{A(x, y_1, \ldots, y_m) \supset C(y_1, \ldots, y_m)}{\exists x A(x, y_1, \ldots, y_m) \supset C(y_1, \ldots, y_m).}$$

Similarly, using $\Lambda p \{\psi((p)_0, y_1, \ldots, y_m)\}((p)_1)$ as realization function for the conclusion, given that ψ is for the premise.

The theorem includes the simple consistency of the intuitionistic formal system of number theory (by using (a) with Γ empty and $1=0$ as the E), as does Theorem 61 (a). The additional interest in Theorem 62 in this connection stems from the different condition on new axioms Γ under which it is shown that the simple consistency is preserved (as we shall discuss further following Theorem 63).

Corollary 1^N. *If* $y_1, \ldots, y_m$ *are distinct variables, and* $A(y_1, \ldots, y_m)$ *is a formula, then* $A(y_1, \ldots, y_m)$ *is realizable, if and only if* $\forall y_1 \ldots \forall y_m A(y_1, \ldots, y_m)$ *is realizable.*

For $A(y_1, \ldots, y_m)$ and $\forall y_1 \ldots \forall y_m A(y_1, \ldots, y_m)$ are interdeducible in the intuitionistic formal system.

This corollary (applied to the case $y_1, \ldots, y_m$ are the free variables of the given formula in order of first free occurrence) gives the equivalence of the present version of the definition of realizability (Kleene 1948) to that of Kleene 1945.

Corollary 2^N. (a) *If* Γ *are realizable formulas,* $A(x_1, \ldots, x_n, y)$ *is a formula containing free only the distinct variables* $x_1, \ldots, x_n, y$, *and* $\Gamma \vdash \exists y A(x_1, \ldots, x_n, y)$ *in the intuitionistic number-theoretic formal system, then there is a general recursive function* $y = \varphi(x_1, \ldots, x_n)$ *such that, for every* $x_1, \ldots, x_n$, $A(x_1, \ldots, x_n, y)$ (*where* $y = \varphi(x_1, \ldots, x_n)$) *is realizable.*

(b) *Similarly reading in place of "realizable" any one of the following combinations of properties*: (i) "*realizable-*($\Gamma \vdash$) *and deducible from* Γ", (ii) "*realizable-*($\Gamma \vdash$), *deducible from* Γ, *and true*", (iii) "*realizable-*($\Gamma \vdash$), *deducible from* Γ, *and realizable*", (iv) "*realizable-*($\Gamma \vdash$), *deducible from* Γ, *true and realizable*".

Proofs. (a) By (a) of the theorem with (B) and (A) 6 of the definitions. (b) (i) Using instead (b) of the theorem. (ii) Using further Theorem 61 (a) to infer that $A(x_1, \ldots, x_n, y)$ is true. (iii) Using further (a) of the theorem to infer that $A(x_1, \ldots, x_n, y)$ is realizable.

Realizability is intended as an intuitionistic interpretation of a formula; and to say intuitionistically that $A(x_1, \ldots, x_n, y)$ is realizable should imply its being intuitionistically true, i.e. that the proposition $A(x_1, \ldots, x_n, y)$ constituting its intuitionistic meaning holds. The formula $\exists y A(x_1, \ldots, x_n, y)$ asserts the existence, for every $x_1, \ldots, x_n$, of a y depending on $x_1, \ldots, x_n$ such that $A(x_1, \ldots, x_n, y)$; or in other words, the existence of a function $y = \varphi(x_1, \ldots, x_n)$ such that, for every $x_1, \ldots, x_n$, $A(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n))$. By (a) of the corollary for Γ empty, that formula can be proved in the intuitionistic formal system, only when there exists such a φ which is general recursive. In brief, only number-theoretic functions which are general recursive can be proved to exist intuitionistically. (We are here considering the assertion of the existence of a function value $\varphi(x_1, \ldots, x_n)$ for all n-tuples $x_1, \ldots, x_n$ of arguments, so this is not in conflict with our use intuitionistically of partial recursive functions.)

This result as inferred from (a) depends on accepting the thesis that the realizability of $A(x_1, \ldots, x_n, y)$ implies its truth. However by using (b) for Γ empty (in which case, since we have no hypothesis on Γ to satisfy, we may take the strongest form (iv) in the conclusion, i.e. that $A(x_1, \ldots, x_n, y)$ is realizable-($\vdash$), provable, true and realizable), we obtain the same result independently of that thesis.

The presence of the Γ in the corollary shows that the result will hold good upon enlarging the formal system by any suitable axioms Γ. If the thesis that realizability implies truth, intuitionistically, is accepted, these need only be realizable. Otherwise they should be realizable-($\Gamma \vdash$) and true (deducibility from Γ holds automatically in the hypothesis on Γ).

The result provides a connection between Brouwer's logic as formalized by Heyting and Church's thesis (§ 62) that only general recursive functions are effectively calculable. Both developments arose from a constructivistic standpoint, but were previously unrelated in their details.

The formula $\exists y A(x_1, \ldots, x_n, y)$ does not assert the uniqueness of the function $y = \varphi(x_1, \ldots, x_n)$ such that $A(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n))$; for this we need $\exists! y A(x_1, \ldots, x_n, y)$ (§ 41).

Classically, given the existence of some function φ such that, for all $x_1, \ldots, x_n$, $A(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n))$, the least number principle provides formally a method of describing a particular one (*149 § 40, *174b § 41). While we do not have the least number principle intuitionistically, we do know by Corollary 2 that, whenever a particular intuitionistic proof of a formula of the form $\exists y A(x_1, \ldots, x_n, y)$ is given, we can on the basis of that proof describe informally a particular general recursive function $\varphi(x_1, \ldots, x_n)$ such that, for all $x_1, \ldots, x_n$, $A(x_1, \ldots, x_n, \varphi(x_1, \ldots, x_n))$.

Example 1. (Cf. Example 8 (c) § 74.) Let S_1 be the intuitionistic number-theoretic system. Let A(x, y) be a formula containing free only x and y. Suppose that for each x, the formula A($\boldsymbol{x}, \boldsymbol{y}$) is true for exactly one y. Then when (to obtain S_2) we introduce f with the axiom A(x, f(x)), the axiom characterizes f as expressing a certain function φ under the interpretation. By $\exists$-introd. from the new axiom, $\vdash_2 \exists y A(x, y)$. Now suppose f with the axiom A(x, f(x)) is eliminable. Then $\vdash_1 \exists y A(x, y)$. Then by Theorem 62 Corollary 2 (b) (ii) with Γ empty, there is a general recursive function $y = \varphi_1(x)$ such that, for each x, A($\boldsymbol{x}, \boldsymbol{y}$) is true. But then $\varphi_1 = \varphi$. Thus *in the intuitionistic number-theoretic system, a new function symbol* f *(expressing a function* φ*) introduced with an axiom of the form* A(x, f(x)), *where* A(x, y) *contains free only* x *and* y, *and* A($\boldsymbol{x}, \boldsymbol{y}$) *is true exactly when* $y = \varphi(x)$, *is eliminable only when* φ *is general recursive.*

Example 2. Let A(x, y) be any formula, containing free only x and y, such that $\vdash_1 \exists y A(x, y)$. Then as in Example 1, there is a general recursive function $y = \varphi_1(x)$ such that, for each x, A($\boldsymbol{x}, \boldsymbol{y}$) is true. The demonstration of this (consisting mainly in the proof of Theorem 62 (b)) is constructive; given a proof of $\exists y A(x, y)$ (or the Gödel number of such a proof), we can find a system E of equations defining a φ_1 recursively (or a Gödel number of φ_1). Also it is effectively decidable whether a number a is the Gödel number of a proof of a formula of the form $\exists y A(x, y)$ where A contains free only x and y (Case 1), or not (Case 2). Let

$$\theta(a) = \begin{cases} \text{a Gödel number of } \varphi_1, \text{ in Case 1}, \\ \Lambda x\, x \text{ (i.e. a Gödel number of } U_1^1), \text{ in Case 2}, \end{cases}$$

where ambiguity as to which Gödel number of which φ_1 (or which Gödel number of U_1^1) is chosen is removed by suitable conventions. Then $\theta(a)$ is effectively calculable. So by Church's thesis we may expect that $\theta(a)$ is general recursive. (In fact, it is easy to prove that $\theta(a)$ is primitive recursive, after establishing: (1) *There is a primitive recursive function* $\xi(a)$ *such that, if a is the Gödel number of a proof in the intuitionistic number-theoretic system, then* $\xi(a)$ *is a Gödel number of a realization-*($\vdash$) *function* $\varphi(y_1, \ldots, y_m)$ *for the endformula* A($y_1, \ldots, y_m$), *where* $y_1, \ldots, y_m$ *are the free variables of the endformula in order of occurrence in our list of the variables.*) Let $\varphi(x) = \{\theta(x)\}(x)+1$. Then $\varphi(x)$ is general recursive. Now let A(x, y) be a formula such that A($\boldsymbol{x}, \boldsymbol{y}$) is true exactly when $y = \varphi(x)$ (e.g. one which numeralwise represents φ, cf. Theorem 32 (a) § 59). If we now take this formula as the A(x, y) of Example 1, we are led to a contradiction by supposing that f with the axiom A(x, f(x)) is eliminable. Thus: (2) *There is a general recursive function* φ *such that, in the intuitionistic number-theoretic system, a new function symbol* f *expressing* φ *with an axiom of the form* A(x, f(x)), *where* A(x, y) *contains free only* x *and* y, *and* A($\boldsymbol{x}, \boldsymbol{y}$) *is true exactly when* $y = \varphi(x)$, *is not eliminable (and* $\exists y$A(x, y) *is not provable for any such* A(x, y)).

Theorem 63[N]. *For suitably chosen formulas* A(x), B(x) *and* C(x, y), *the following classically provable formulas are unrealizable and hence (by Theorem 62 (a)) unprovable in the intuitionistic formal system of number theory. (Specifically, let* A(x, z) *numeralwise express the predicate* $T_1(x, x, z)$ *of* § 57, using Corollary Theorem 27 § 49. *Let* A(x) *be* $\exists z$A(x, z), B(x) *be* A(x) $\vee \neg$A(x) *and* C(x, y) *be* y=1 $\vee$ (A(x) & y=0).)

(i) $A(x) \vee \neg A(x)$.

(ii) $\forall x(A(x) \vee \neg A(x))$ (the closure of (i)).

(iii) $\neg\neg\forall x(A(x) \vee \neg A(x))$ (the double negation of (ii)).

(iv) $\forall x \neg\neg B(x) \supset \neg\neg\forall x B(x)$.

(v) $\neg\neg\{\forall x \neg\neg B(x) \supset \neg\neg\forall x B(x)\}$ (the double negation of (iv)).

(vi) $\exists y C(x, y) \supset \exists y[C(x, y) \,\&\, \forall z(z<y \supset \neg C(x, z))]$ (cf. *149 § 40).

(vii) $\exists y[y<w \,\&\, C(x, y) \,\&\, \forall z(z<y \supset \neg C(x, z))] \vee \forall y[y<w \supset \neg C(x, y)]$ (cf. *148).

Also the closure, and the double negation of the closure, of (vi) *and of* (vii). ((i) — (v): Kleene 1945 with Nelson 1947.)

Lemma 46[N]. (a) *If* A *is realizable, and* B *is unrealizable, then* A $\supset$ B *is unrealizable. Hence: If* A *is realizable, then* $\neg$A *is unrealizable.* (b) *If* A *is closed and unrealizable, then* A $\supset$ B *and* (hence) $\neg$A *are realizable, and* (by (a)) $\neg\neg$A *is unrealizable.*

PROOF OF LEMMA 46. (a) By Theorem 62 (a) or the case for Rule 2 in its proof, if A and A ⊃ B are realizable, so is B. (b) For a closed B, any number, e.g. 0, realizes A ⊃ B, since vacuously, whenever a realizes A (i.e. never), $0(a)$ realizes B.

LEMMA 47[N]. *If* $\mathrm{P}(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ *numeralwise expresses a general recursive predicate* $P(x_1, \ldots, x_n)$ *in the intuitionistic formal system of number theory, then, for every* $x_1, \ldots, x_n$, $\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ *is realizable if and only if* $P(x_1, \ldots, x_n)$.

PROOF OF LEMMA 47. If $P(x_1, \ldots, x_n)$, then by § 41 (i), $\vdash \mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, and hence by Theorem 62 (a), $\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is realizable. Conversely, suppose $\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is realizable. Because $P(x_1, \ldots, x_n)$ is a general recursive predicate, we have (constructively) that, for the given $x_1, \ldots, x_n$, either $P(x_1, \ldots, x_n)$ or $\overline{P}(x_1, \ldots, x_n)$. In the latter case, however, by § 41 (ii), $\vdash \neg\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, and hence by Theorem 62 (a), $\neg\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is realizable, which by Lemma 46 (a) contradicts our supposition that $\mathrm{P}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is realizable.

PROOF OF THEOREM 63. (i) Suppose (i), i.e. ∃zA(x, z) V ¬∃zA(x, z), were realizable. Let $\varphi(x)$ be a realization function for it; and set $\rho(x) = (\varphi(x))_0$. Then $\rho(x)$ is general recursive, and takes only the values 0 and 1 (by (B) and (A) 3 of the definitions). Consider any fixed x. CASE 1: $\rho(x) = 0$. Then $(\varphi(x))_1$ realizes ∃zA($\boldsymbol{x}$, z); and hence $(\varphi(x))_{1,1}$ realizes A($\boldsymbol{x}$, $\boldsymbol{z}$) where $z = (\varphi(x))_{1,0}$, in which case by Lemma 47, $T_1(x, x, z)$. Thus $(Ez)T_1(x, x, z)$. CASE 2: $\rho(x) = 1$. Then $(\varphi(x))_1$ realizes ¬∃zA($\boldsymbol{x}$, z), i.e. $(\varphi(x))_1$ realizes ∃zA($\boldsymbol{x}$, z) ⊃ 1=0. We shall show that then $\overline{(Ez)}T_1(x, x, z)$. For if there were a z such that $T_1(x, x, z)$, by Lemma 47 A($\boldsymbol{x}$, $\boldsymbol{z}$) would be realizable; say k realizes it. Then $2^z \cdot 3^k$ would realize ∃zA($\boldsymbol{x}$, z); and $\{(\varphi(x))_1\}(2^z \cdot 3^k)$ would realize 1=0, which is impossible. The two cases show that the general recursive function $\rho(x)$ is the representing function of $(Ez)T_1(x, x, z)$. But $(Ez)T_1(x, x, z)$ is non-recursive ((15) Theorem V § 57); hence no such general recursive $\rho(x)$ can exist. By reductio ad absurdum, therefore (i) is unrealizable.

(ii), (iii). By ∀-elim. (i) is deducible intuitionistically from (ii), so by Theorem 62 (a) also (ii) is unrealizable; and by Lemma 46 (b) so is (iii), since (ii) is closed.

(iv) Since (iii) can be deduced from (iv), using *51a § 27 and ∀-introd.

(vi) We show as follows that (i) is deducible from (vi). From 1=1 (which is provable) by V- and ∃-introd., ∃yC(x, y). Using this, from (vi) by ⊃-elim., ∃y[C(x, y) & ∀z(z<y ⊃ ¬C(x, z))]. Preparatory to &- and

∃-elim., assume C(x, y), i.e.

(1) y=1 V (A(x) & y=0)

and ∀z(z<y ⊃ ¬C(x, z)), i.e.

(2) ∀z(z<y ⊃ ¬{z=1 V (A(x) & z=0)}).

We use proof by cases from (1) to deduce (i) with the help of (2). CASE 1: assume y=1. For reductio ad absurdum, assume further A(x). From this and 0=0, by &- and V-introd., 0=1 V (A(x) & 0=0). But also from y=1 by *135b, 0<y; and thence from (2) by ∀-elim. (with 0 as the t) and ⊃-elim., ¬{0=1 V (A(x) & 0=0)}. Hence by reductio ad absurdum, ¬A(x). By V-introd., A(x) V ¬A(x), which is (i) and does not contain free the variable y of our proposed ∃-elim. CASE 2: assume A(x) & y=0. By &-elim. and V-introd., A(x) V ¬A(x). (This deduction is related to the intuitive reasoning of Example 6 § 64.)

(vii) From (vii) we can deduce (vi), as in the proof of *149 from *148.

Theorem 63 (i) — (v) imply that $\mathcal{A}$ V ¬$\mathcal{A}$ is unprovable in the intuitionistic propositional calculus, and ∀x($\mathcal{A}$(x) V ¬$\mathcal{A}$(x)), ¬¬∀x($\mathcal{A}$(x) V ¬$\mathcal{A}$(x)), ∀x¬¬$\mathcal{A}$(x) ⊃ ¬¬∀$x$$\mathcal{A}$($x$) and ¬¬{∀$x$¬¬$\mathcal{A}$($x$) ⊃ ¬¬∀$x$$\mathcal{A}$($x$)} in the intuitionistic predicate calculus, as we already knew from Theorem 57 (b) and Theorem 58 (a) and (c). The present proofs are less elementary than those based on Gentzen's normal form theorem, but contribute insight into the working of the intuitionistic logic as an instrument for number-theoretic reasoning. We succeed in showing A V ¬A unprovable in intuitionistic number theory only in the presence of a free variable x.

COROLLARY (to (ii))[N]. *The formula* ¬∀x(A(x) V ¬A(x)) *(although the negation of a classically provable formula) is realizable.*

By (ii) and Lemma 46 (b).

The formula ∀x(A(x) V ¬A(x)) is classically provable, and hence under classical interpretations true. But it is unrealizable. So if realizability is accepted as a necessary condition for intuitionistic truth, it is untrue intuitionistically, and therefore unprovable not only in the present intuitionistic formal system, but by any intuitionistic methods whatsoever.

This incidentally implies that our classical formal system reinforced by an intuitionistic proof of simple consistency cannot serve as an instrument of intuitionistic proof, as suggested in § 14, except of formulas belonging to a very restricted class (including those of the forms B($\boldsymbol{x}$) and ∀xB(x) end § 42, but not the present formula ∀x(A(x) V ¬A(x))).

The negation $\neg\forall x(A(x) \vee \neg A(x))$ of that formula is classically untrue, but (by the corollary) realizable, and hence intuitionistically true, if we accept realizability (intuitionistically established) as sufficient for intuitionistic truth.

So the possibility appears of asserting the formula $\neg\forall x(A(x) \vee \neg A(x))$ intuitionistically. Thus we should obtain an extension of the intuitionistic number theory, which has previously been treated as a subsystem of the classical, so that the intuitionistic and classical number theories diverge, with $\neg\forall x(A(x) \vee \neg A(x))$ holding in the intuitionistic and $\forall x(A(x) \vee \neg A(x))$ in the classical.

Such divergences are familiar to mathematicians from the example of Euclidean and non-Euclidean geometries, and other examples, but are a new phenomenon in arithmetic. The first example comes by adjoining $A_p(p)$ or $\neg A_p(p)$ to the number-theoretic formalism, cf. end §§ 42 and 75.

Not only is the formula $\neg\forall x(A(x) \vee \neg A(x))$ itself realizable, but by Theorem 62 (a) (taking it as the Γ), when we add it to the present intuitionistic formal system, only realizable formulas become provable in the enlarged system. So then every provable formula will be true under the realizability interpretation. In particular, the strengthened intuitionistic system is thus shown by interpretation to be simply consistent.

A fuller discussion is given in Kleene 1945, where the proposed adjunctions to the unstrengthened intuitionistic formal system of number theory S, to obtain a strengthened intuitionistic system S' diverging from the classical S_c, are in the form of an identification of truth with realizability.

Refinements of the results which we are basing here on interpretation are obtained by Nelson 1947 Parts II—IV (with Kleene 1945). Because they all involve the consistency of the number-theoretic formalism, no completely elementary treatment can be expected. But the non-elementariness is minimized in the results based on this further work of Nelson to the full extent that the results are proved in elementary metamathematics under the hypothesis of the simple consistency of S. In particular, by these results with those of Gödel 1932-3 (cf. Corollary 2 Theorem 60), it is demonstrated metamathematically that both S' and S_c are simply consistent if S is. (Nelson takes as his S not our intuitionistic formal system but one obtained, apart from an inessential difference in the equality postulates, by adjoining to ours some additional function symbols with their defining equations. These equations fit our schemata (I) — (V) § 43 or closely similar schemata, except that also a certain schema of course-of-values recursion is allowed. Using Nelson's (i) — (iv) p. 332, to each application of that schema a pair of equations having the same form with f, g, h, t_i replaced by f′, g′, h′, t_i' is provable without the application; so the course-of-values recursion schema is eliminable. Then by Example 9 § 74 with the remarks preceding it, the additional function symbols are eliminable.)

Nelson 1949 introduces a notion of 'P-realizability', using which one can set up a number-theoretic system diverging from both the strengthened intuitionistic and the classical.

Gene Rose 1952 investigates realizability in relation to the intuitionistic propositional calculus.

Kleene 1950a plans the use of recursive functions in interpreting intuitionistic set theory.

Example 3. (a) *The operators* $\supset, \neg, \&, \vee$ *applied to closed formulas* A *and* B *obey the strong* 3-*valued truth tables* (§ 64, restated with the present symbols), *when* t, f, u *are read as 'realizable', 'unrealizable', 'unknown (or value immaterial)', respectively*; i.e. the tables then give only correct information about the realizability or unrealizability of $A \supset B$, $\neg A$, A & B, $A \vee B$, when entered from such information about A and B. Proof. Consider $\supset$. If B is realizable, then by *11 § 26 with Theorem 62 (a), so is $A \supset B$, corresponding to the three t's in Column 1 of the table for $\supset$. If A is unrealizable, then by Lemma 46 (b), $A \supset B$ is realizable, corresponding to the three t's in Row 2. If A is realizable and B is unrealizable, then by Lemma 46 (a), $A \supset B$ is unrealizable, corresponding to the f in Row 1 Column 2. The table for $\neg$ is simply the f column of that for $\supset$; and & and $\vee$ are easily treated. (b) *A formula without variables is realizable, if and only if it is true.* Its realizability (and truth) or unrealizability (and falsity) is thus effectively decidable by the valuation procedure furnished by the usual interpretation of 0, ′, +, ·, = and the classical 2-valued truth tables for $\supset, \neg, \&, \vee$ (cf. § 79 before Theorem 51). Proof by using Example 4 § 81, or thus: For closed prime formulas, truth and realizability agree, and can be decided. In building thence composite formulas by the operations of the propositional calculus, we always remain within the first two rows and columns of the 3-valued tables. (c) We call a number e an R-*valuation number* of a closed formula E, if either $e = 2^0 \cdot 3^{e_1}$ (then $e_1 = (e)_1$) and e_1 realizes E, or $e = 2^1 \cdot 3^0$ and E is unrealizable. For open formulas, R-*valuation function* is defined in analogy to 'realization function'. *A formula* $C(z_1, \ldots, z_m)$ *containing no quantifiers and only the distinct variables* $z_1, \ldots, z_m$ $(m > 0)$ *has a primitive recursive* R-*valuation function* $\gamma(z_1, \ldots, z_m)$. Proof (omitting "$z_1, \ldots, z_m$"